Joyal Model Structure

$U_n : \partial\Delta^n \longrightarrow \Delta^n$

$V^n_k : \Lambda^n_k \longrightarrow \Delta^n$

$\omega : \Delta^0 \longrightarrow J$

Def^n: $f : X \to Y$ is inner anodyne if $f \boxslash$ inner fibrations.

$\hat{\times}$ - pushout product

$\triangleright$ - pullback exponential.

Def^n's: $f : X \to Y$ is called a cat. htpy eq. if $\exists g : Y \to X$ and $H : fg \simeq id$ and $G : gf \simeq id$.

Def^n: $f : X \to Y$ is called a cat eq. if for any ∞-cat K, $K^Y \to K^X$ is a cat htpy eq.

Rmk: cat homtpy eq. $\Leftrightarrow$ cat eq. b/w ∞-cats.

Joyal Structure:

1) cof : mono
2) w.e : cat eq.
3) fib. obj : ∞-cats.

(monos, acyclic kan fib) $\to$ Quillen
(acyclic cof, fibs).

Outline:
1) Technical Lemmas
2) Comparison kan fib vs. fibs (Joyal)
3) Acyclic cofs $\to$ find gen. set.
4) Show model structure exists.

Lemma:
1) $U_m \hat{\times} V^2_1$ is inner anodyne.
2) V^n_k is a retract $V^n_k \hat{\times} V^2_1$

Prop: If i mono, p inner fib, then $i \triangleright p$ is an inner fibration.

Pf: $V^n_k \boxslash (i \triangleright p) \Leftarrow V^n_k \hat{\times} V^2_1 \boxslash (i \triangleright p)$
0<k<n
$\Leftrightarrow i \hat{\times} V^n_k \boxslash (V^2_1 \triangleright p)$
$\Leftarrow U_m \boxslash (V^2_1 \triangleright p) \Leftrightarrow \underbrace{(U_m \hat{\times} V^2_1)}_{\text{inner anodyne.}} \boxslash p$. $\square$.

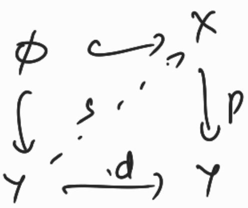
Prop: If i mono, p is an inner isofib b/w ∞-cats, then $i \triangleright p$ is an inner isofib.

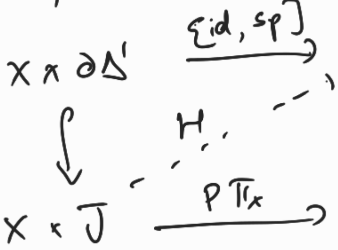
Lemma: If $p : X \to Y$ b/w ∞-cats which is a inner isofib + cat eq. Then, $\exists\ s : Y \to X$ deformation section.

f $s: Y \to X$ ...

Prop: If $p: X \to Y$ b/w ∞-cats. p inner isofib + cat eq. $\iff$ acyclic Kan fib.

Pf: $(\Leftarrow)$ p acyclic Kan. $ps = id_Y$





$H: id \simeq sp$

$\Rightarrow p$ is a cat htpy eq.

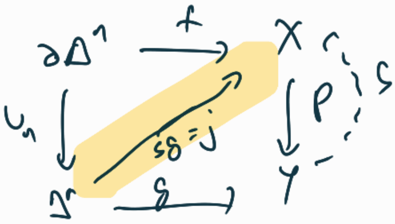
$\Rightarrow p$ is a cat eq.

Note: Don't need X, Y to be ∞-cats.

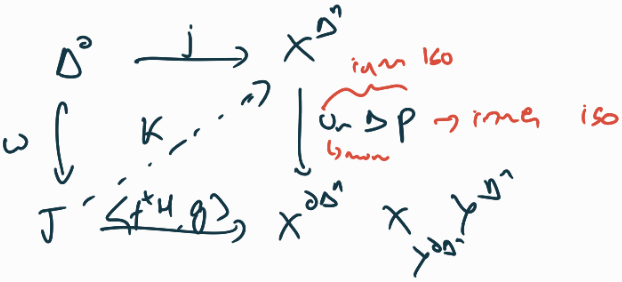
p inner isofib ✓.

$(\Rightarrow)$ p inner isofib + cat eq. $ps = id_Y$, $H: sp \simeq id_X$, $pj = psg = g$

$j u_n = s g u_n = spf \simeq f$ by $f^* H$



$K|_{\{1\}} = h$, $K: j \simeq h$



$ph = g$. ✓

$h u_n = K|_{\{1\}} u_n = u_n^* K|_{\{1\}}$

$= f^* H|_{\{1\}} = f$. ✓

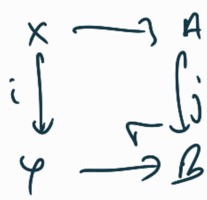
$\Rightarrow p$ is a acyclic Kan fibration.

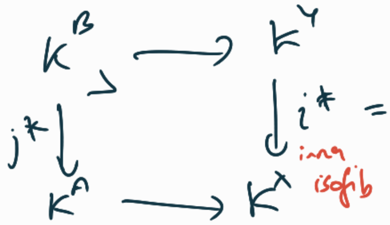
Acyclic cofibrations: monos + cat. eq.

Fact: Λ^n_k, $0<k<n$ and w are acyclic cof.

Prop^n: acyclic cof form a saturated class.

Pf: (Pro hard) $i: X \hookrightarrow Y$, j mono ✓ Let K be any ∞-cat





$i^* = i \rhd (K \to \Delta^0) = K^Y \to K^X$ $\quad X(\Delta^0)^Y$, $(\Delta^0)^X$

mono inner isofib

i cat eq $\Rightarrow i^*$ is a cat htp eq. $\Rightarrow i^*$ is a cat eq.

$\Rightarrow i^*$ is acyclic Kan $\Rightarrow j^*$ is acyclic Kan $\Rightarrow j^*$ cat eq

$\Rightarrow j^*$ cat htpy eq.

$\Rightarrow j$ is a cat eq. D.

$R\{\Lambda^1_2 \cup \omega\}$ gives us map that we don't want to be fibrations.

{acyc cof}.

$S = \{$acyc cofs b/w ctble sSets$\}$

Prop: Bounded acyc cof. Lemma: Given $i : X \hookrightarrow Y$ acyc cof. and $A \underset{\text{ctble}}{\subseteq} Y$, $\exists B \underset{\text{ctble}}{\subseteq} Y$, $A \subseteq B$ s.t. $B \cap X \hookrightarrow B$.

(Pf Idea) $A \cap X \xhookrightarrow{j_A} A$, j_A cat eq $\Leftrightarrow RQj_A$ acyclic Kan fib.

A', j_A " A' solves this lifting problem for A"

Pf: Run SOA on inner horns to get $Q : K \longmapsto$

$$\begin{array}{ccc} K & \longrightarrow & \Delta^0 \\ & \searrow^{\sim} \quad \nearrow_{\text{inner fib}} & \\ & QK & \end{array}$$

$$R : (K \xrightarrow{f} L) \mapsto \begin{array}{ccc} K & \xrightarrow{f} & L \\ & \searrow^{\sim} \quad \nearrow_{Rf} & \\ & K_f & \end{array} \quad Rf = \text{inner isofib}$$

Fact: R and Q preserve filtered colimits.

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow\wr & & \downarrow\wr \\ QK & \xrightarrow{Qf} & QL \\ & \searrow^{\wr} \quad \nearrow_{RQf} & \\ & R_{Qf} & \end{array}$$

f cat eq $\Leftrightarrow Qf$ cat eq.
$\Leftrightarrow RQf$ cat eq
$\Leftrightarrow RQf$ acyclic Kan.

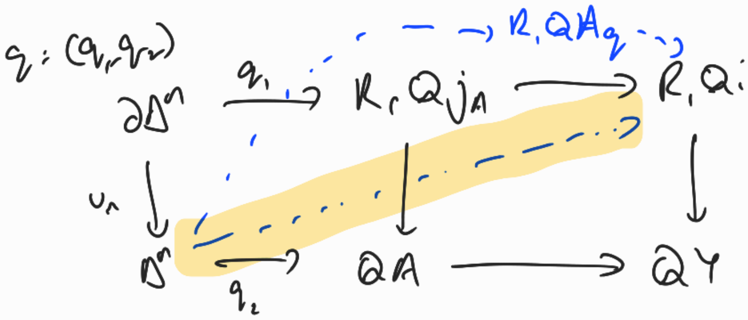
$C \underset{\text{ctble}}{\subseteq} Y$

$$\begin{array}{ccc} C \cap X & \longrightarrow & X \\ j_C \downarrow & & \downarrow i \\ C & \longrightarrow & Y \end{array}$$

$$\begin{array}{ccc} \operatorname{colim}_{C \underset{\text{ctble}}{\subseteq} Y} C \cap X & \xrightarrow{\cong} & X \\ \downarrow & & \downarrow i \\ \operatorname{colim} C & \xrightarrow{\cong} & Y \end{array}$$

$$\begin{array}{ccc} \operatorname{colim} Q(C \cap X) & \xrightarrow{\cong} & QX \\ \downarrow & & \downarrow Qi \\ \operatorname{colim} QC & \xrightarrow{\cong} & QY \end{array}$$

$$\begin{array}{ccc} \operatorname{colim} Q(C \cap X) & \xrightarrow{\cong} & QX \\ \downarrow & & \downarrow \\ \operatorname{colim} R_{Q j_C} & \xrightarrow{\cong} & RQi \end{array}$$

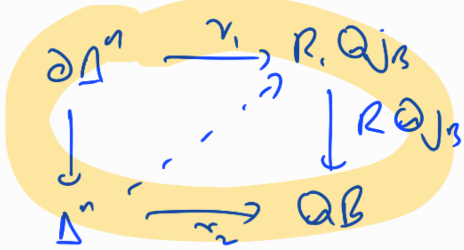


$A' := \bigcup_q A_q \cup A$, $A \subseteq A'$

$A_0 := A$

$A_{n+1} := A_n'$

$B = \bigcup_n A_n$



RQj_B is acyclic Kan $\Rightarrow$ $j_B : B \cap X \hookrightarrow B$ is cat. eq.

$R(S) =:$ fibs

Prop: (acyclic cof, fibs) form a wfs.

Pf: Enough to show: $i : X \xrightarrow{\sim} Y$ can be generated using S
i.e, i is an S-cell complex.

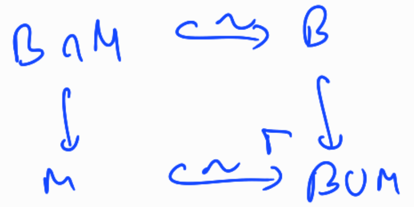
$P = \{K \subseteq Y \mid X \to K$ is an S-cell complex$\}$

$\leq_P$, $K \leq_P L$, $K \to L$ is an S-cell complex.

Zorn's: M max'l in P.

$M \neq Y$, $\exists y \in Y \setminus M$, $\exists A \subseteq Y$ ctble s.t. $y \in A$.

$\exists B \subseteq Y$ ctble, $A \subseteq B$



Contradicts max'l of M.

$\Rightarrow M = Y$. Thus $i : X \xrightarrow{\sim} Y$ is an S-cell complex. D.

Th^m: We have a model structure
1) cof: monos.
2) w.e: cat eq

Pf: 2 wfs (cof, acyclic Kan fibration)
(acyclic cofs, fibs)

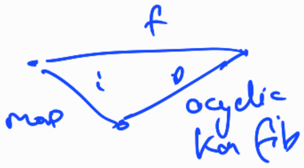
Enough to show: fib + cat eq. $\iff$ acyclic Kan fib.

Pf: ($\Leftarrow$) fib rlp acyc. cofs b/w cblk sSets

$\Rightarrow$ acyclic Kan fib. are fib (Joyal)

acyclic Kan fib is a cat. eq (by prev. prop)

($\Rightarrow$) f fib + cat. eq.



2-out-of-3 $\Rightarrow$ i cat eq.

$i \boxslash f$

$\Rightarrow$ f is a retract of p

$\Rightarrow$ f is an acyclic Kan fib. ∎